SPRING 2025: MATH 590 EXAM 2

You must show all work to receive full credit. No calculators or notes allowed.

Name:

Throughout the exam, all vector spaces are finite dimensional and are defined either over $\mathbb{R}$ or $\mathbb{C}$.

(I) True-False. Write true or false next to each question. You do not have to justify your answer. (2 points each)

(i) If A is a unitarily diagonalizable square matrix over $\mathbb{C}$, then A is self adjoint. False.Such a matrix is normal, but not necessarily self-adjoint.

(ii) For A an $n \times n$ matrix over $\mathbb{R}$ and $v, w \in \mathbb{R}^n$, $\langle A^t v, w\rangle = \langle v, Aw\rangle$. True. $\langle A^t v, w\rangle = \langle v, A^{tt} w\rangle = \langle v, Aw\rangle$

(iii) If A is an $n \times n$ normal matrix over $\mathbb{C}$, then A has its eigenvalues in $\mathbb{R}$. False, as exhibited by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(iv) Any self-adjoint complex matrix is a normal matrix. True.

(v) Suppose A is a 2×2 diagonalizable matrix with $p_A(x) = (x-7)^2$. Then $A = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$. True. If $P^{-1}AP = 7 \cdot I_2$, then $A = P(7 \cdot I_2)P^{-1} = 7 \cdot PI_2P^{-1} = 7 \cdot I_2$.

(II) Statements. State the following theorems. **Define all relevant terms** in each of the statements (but you do not have to define inner product). (5 points each)

1. State the theorem characterizing when a matrix is diagonalizable.

Solution. The matrix A is diagonalizable if and only if $p_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ and $\dim(E_{\lambda_1}) = e_i$, for all $1 \leq i \leq r$.

Here: E_{λ_i} denotes the eigenspace of the eigenvalue λ_i.

2. State the theorem describing the Gram-Schmidt process as it applies to the set $\{v_1, v_2, v_3\}$ of linearly independent vectors.

Solution. An orthogonal set w_1, w_2, w_3 satisfying $\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{w_1, w_2, w_3\}$ is obtained as follows:

$$\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 \\
w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} \cdot w_2
\end{aligned}$$

Note: The vectors w_1, w_2, w_3 are an orthogonal set if $\langle w_i, w_j \rangle = 0$, for $i \neq j$.

3. State the Complex Spectral Theorem for matrices.

Solution. A complex matrix is normal if and only if it is unitarily diagonalizable.

Note: A is normal if, $AA^* = A^*A$, where A^*, the adjoint of A, is the conjugate transpose of A and the matrix diagonal Q is unitary if $Q^{-1} = Q^*$.

4. State the Singular Value Decomposition Theorem

Solution. Given and $m \times n$ matrix over $\mathbb{R}$, there exist orthogonal matrices Q and P and an $m \times n$ diagonal matrix $\sum$ such that $A = Q \sum P^t = Q \sum P^{-1}$. The non-zero diagonal entries of $\sum$ are arranged as $\sigma_1 \geq \cdots \geq \sigma_r$, and are called the singular values of A, and occur as the square roots of the non-zero eigenvalues of $A^T A$.

A matrix P over $\mathbb{R}$ is orthogonal if $P^t = P^{-1}$.

(III) Calculation problems. (15 points each)

1. For $A = \begin{pmatrix} 2 & 0 & 2i \\ 0 & 1 & 0 \\ -i & 0 & 3 \end{pmatrix}$, determine if A is diagonalizable, unitarily diagonalizable, or not diagonalizable. If A is diagonalizable or unitarily diagonalizable, find the appropriate diagonalizing matrix P, but you do not have to check the $P^{-1}AP$ product.

Solution. $AA^* = \begin{pmatrix} 8 & 0 & 5i \\ 0 & 1 & 0 \\ -8i & 0 & 10 \end{pmatrix}$, $A^*A = \begin{pmatrix} 5 & & \\ & & \\ & & \end{pmatrix}$, so $AA^* \neq A^*A$, and thus A is not normal, and therefore not unitarily diagonalizable.

$p_A(x) = \begin{vmatrix} x-2 & 0 & -2i \\ 0 & x-1 & 0 \\ i & 0 & x-3 \end{vmatrix} = (x-1)\{(x-2)(x-3)-2\} = (x-1)^2(x-4)$. Thus, $\lambda = 1, 4$ are the eigenvalues of A.

$E_1 =$ null space of $\begin{pmatrix} 1 & 0 & 2i \\ 0 & 0 & 0 \\ -i & 0 & 2 \end{pmatrix} \overset{\text{EROs}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 2i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so E_1 has basis $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2i \\ 0 \\ -1 \end{pmatrix}$.

$E_4 =$ null space of $\begin{pmatrix} -2 & 0 & 2 \\ 0 & -3 & 0 \\ -i & 0 & -1 \end{pmatrix} \overset{\text{EROs}}{\longrightarrow} \begin{pmatrix} i & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so E_4 has basis $\begin{pmatrix} -1 \\ 0 \\ i \end{pmatrix}$. Since the algebraic multiplicity equals the geometric multiplicity for each eigenvalue, P is diagonalizable, with diagonalizing matrix $\begin{pmatrix} 0 & 2i & -1 \\ 1 & 0 & 0 \\ 0 & -1 & i \end{pmatrix}$.

2. Find the singular value decomposition for the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix}$. **Verify that your decomposition works.**

Solution. $A^tA = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$. $p_{A^tA}(x) = (x-5)^2 - 9 = (x-8)(x-2)$. Thus, the eigenvalues of A^tA are $8, 2$, so the singular values of A are $\sqrt{8}, \sqrt{2}$. In particular, $\sum = \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$. To find u_1, u_2, we orthogonally diagonalize A^tA.

$E_8 =$ null space of $\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \overset{\text{EROs}}{\longrightarrow} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, so E_8 has orthogonal basis $u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

$E_2 =$ null space of $\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \overset{\text{EROs}}{\longrightarrow} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, so E_2 has orthogonal basis $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Thus, we take $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. For the columns of Q, we calculate

$v_1 = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. If we take $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, we get an orthonormal basis for $\mathbb{R}^3$. Therefore, we take $Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, so that $A = Q\sum P^t$, is the singular value decomposition of A.

To verify this: $Q\sum P^t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \\ \sqrt{8} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix}$.

3. Given $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, verify that A is a normal and find a unitary matrix Q that diagonalizes A. **You do not have to verify that the matrix Q works.**

Solution. $AA^t = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = A^tA$, so A is a normal matrix. $p_A(x) = (x-1)^2 + 4 = x^2 - 2x + 5$, which has roots $1 \pm 2i$.

$E_{1+2i} =$ nullspace of $\begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$, so an orthonormal basis for E_{1+2i} is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$.

$E_{1-2i} =$ nullspace of $\begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$, so an orthonormal basis for E_{1-2i} is $\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$. Thus, the matrix $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ unitarily diagonalizes A.

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(IV) Proof Problem. State the general form of the real spectral theorem, and then prove the theorem for 2×2 matrices. (25 points)

Solution. The Spectral Theorem for real matrices states that a matrix with entries in $\mathbb{R}$ is symmetric if and only if it is orthogonally diagonalizable.

Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. We first note that A has real eigenvalues. For this, $p_A(x) = (x-a)(x-c) - b^2 = x^2 - (a+c)x + (ac - b^2)$. To see that this polynomial has real roots, we just have to see that the discriminant, $(a+c)^2 - 4(ac - b^2 \geq 0$. But this is easily seen to be $(a-c)^2 + b^2$, which is always greater than or equal to zero. Note that if the discriminant equals zero, $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, and there is nothing to do.

Assuming the discriminant is non-zero, $p_a(x)$ has two distinct real roots, λ_1, λ_2, the eigenvalues of A. Take $v_1 \in E_{\lambda_1}$ and $v_2 \in E_{\lambda_2}$. Then,

$$\lambda_1\langle v_1, v_2\rangle = \langle \lambda_1 v_1, v_2\rangle = \langle Av_1, v_2\rangle = \langle v_1, Av_2\rangle = \langle v_1, \lambda_2 v_2\rangle = \lambda_2\langle v_1, v_2\rangle.$$

Here we used the fact that $\langle Av_1, v_2\rangle = \langle v_1, Av_2\rangle$, since A is symmetric. Thus, $\lambda_1\langle v_1, v_2\rangle = \lambda_2\langle v_1, v_2\rangle$. Since $\lambda_1 \neq \lambda_2$, we must have $\langle v_1, v_2\rangle = 0$, so that v_1 is orthogonal to v_2. If we now take $u_1 = \frac{1}{||v_1||} \cdot v_1$ and $u_2 = \frac{1}{||v_2||} \cdot v_2$, then u_1, u_2 is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A, which is what we want.

For the converse, suppose that A is orthogonally diagonalizable, i.e., $P^{-1}AP = D$, where D is diagonal and P is orthogonal, i.e., $P^{-1} = P^t$. Then $A = PDP^t$. Thus,

$$A^t = (PDP^t)^t = P^{tt}D^tP^t = PDP^t = A,$$

since $D = D^t$, for diagonal matrices. Therefore, A is symmetric.